Exercises week 4: Fiber optics and waveguides

Problem 1: Numerical aperture (NA) of a slab waveguide. Show that the numerical aperture for a dielectric slab waveguide with refractive indices \( n_1 \) for the core and \( n_2 \) for the cladding (\( n_1 > n_2 \)) is:

\[
NA = \sqrt{n_1^2 - n_2^2}
\]

Problem 2: Mode shape in a mirror waveguide. Demonstrate the characteristic sine and cosine shape (FoP 8.1-6) of the modes in a mirror waveguide by calculating the interference between two plane waves that propagate at \( \pm \theta_m \).

Problem 3: Wavenumber in an air-filled mirror waveguide. Show that the wavenumber of the mode with index \( m \) in a mirror waveguide has the following form:

\[
\beta_m = k_0 \sqrt{1 - \left( \frac{m \omega_j}{\omega} \right)^2}
\]

Also show that the group velocity for mode \( m \) can be written as follows:

\[
\frac{d\omega}{d\beta_m} = v_{g,m} = \frac{c_0 \beta_m}{\omega}
\]

Problem 4: Calculate the number of modes for a wavelength of 800 nm for a mirror waveguide with separation \( d = 10 \mu m \). Compare with a dielectric waveguide with refractive indices \( n_1 = 1.5 \) (core) and \( n_2 = 1.45 \) (cladding). What are angles in respect to the boundary for the modes with the largest mode index \( m \) in both cases?

Problem 5 (FoP 9.1-2): Numerical aperture of a graded-index fiber. Compare the numerical apertures (NA) of a step-index fiber with \( n_1 = 1.45 \) and \( \Delta = 0.01 \) and a graded-index fiber with \( n_1 = 1.45 \) and \( \Delta = 0.01 \), and a parabolic refractive index profile (\( p = 2 \)). (See Exercise 1.3-2)

Problem 6 (FoP 9.2-1): Modes. A step-index fiber has radius \( a = 5 \mu m \), core refractive index \( n_1 = 1.45 \), and fractional refractive-index change \( \Delta = 0.002 \). Determine the shortest wavelength \( \lambda_c \) for which the fiber is a single-mode waveguide. If the wavelength is changed to \( \lambda_c/2 \), identify the indexes \((l,m)\) of all the guided modes.

Problem 7 (FoP 9.2-3): Propagation constants and group velocities. A step-index fiber with refractive indexes \( n_1 = 1.444 \) and \( n_2 = 1.443 \) operates at \( \lambda_0 = 1.55 \mu m \). Determine the core radius at which the fiber \( V \) parameter is 10. Use Fig. 9.2-3 to estimate the propagation constants of all the guided modes with \( l = 0 \). If the core radius is now changed so that \( V = 4 \), use Fig. 9.2-8 to determine the phase velocity, the propagation constant, and the group velocity of the LP\(_{01}\) mode. Ignore the effect of material dispersion.
Solutions

Problem 1: The numerical aperture of a dielectric slab waveguide is determined by the refractive indexes of the core and the cladding. The critical angle for total internal reflection is defined by \(\sin(\alpha_c) = n_2 / n_1 = \cos(\theta_c)\), whereas \(\alpha_c\) and \(\theta_c\) are the angles in respect to the normal and the boundary, respectively. We find the NA by looking for the largest angle \(\alpha\) that can be coupled from outside into the waveguide and still is being guided by total internal reflection:

\[
\sin(\alpha) = n_1 \sin(\theta_c) = n_1 \sqrt{1 - \cos^2(\theta_c)} = n_1 \sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{n_1^2 - n_2^2}
\]

Problem 2: We calculate the interference between the waves going at angles \(\theta_m\) and \(-\theta_m\). See FoP p. 294 for details.

\[
E_{+\theta_m} + E_{-\theta_m} = A_m (\exp(-jk_{my}y - j\beta_m z) + \exp(j(m-1)\pi)\exp(jk_{my}y - j\beta_m z)) \hat{e}_x
\]
\[
\Leftrightarrow E_{+\theta_m} + E_{-\theta_m} = A_m (\exp(-jk_{my}y) + \exp(j(m-1)\pi + jk_{my}y))\exp(-j\beta_m z) \hat{e}_x
\]
\[
\Leftrightarrow E_{+\theta_m} + E_{-\theta_m} = A_m (\cos(k_{my}y) - j \sin(k_{my}y) + \cos(k_{my}y) + j \sin(k_{my}y), m = 1,3,5,...) \exp(-j\beta_m z) \hat{e}_x
\]
\[
\Leftrightarrow E_{+\theta_m} + E_{-\theta_m} = A_m (2 \cos(k_{my}y), m = 1,3,5,... - 2j \sin(k_{my}y), m = 2,4,6,...) \exp(-j\beta_m z) \hat{e}_x
\]

One can obtain expression 8.1-6 for the shape of the symmetric and asymmetric modes by normalizing the modes. Partial integration yields \(\int_{-d/2}^{d/2} \cos^2(k_{my}y) dy = \int_{-d/2}^{d/2} \sin^2(k_{my}y) dy = \frac{d}{2}\) for \(m=1,3,5,...\) and \(m=2,4,6,...\) respectively.

Problem 3: The wavenumber along the direction of the waveguide can be written as:

\[
\beta_m = \sqrt{k_0^2 - k_{T,m}^2}
\]

whereas \(k_{T,m} = k_0 \sin(\theta_m)\). From there, it follows:

\[
\Rightarrow \beta_m = k_0 \sqrt{1 - \sin^2(\theta_m)} = \frac{m\lambda}{2d}
\]
\[
\Rightarrow \beta_m = k_0 \sqrt{1 - \left(\frac{m\lambda}{2d}\right)^2}, \quad \lambda_c = 2d, \quad \lambda = \frac{2\pi c}{\omega}
\]
\[
\Rightarrow \beta_m = k_0 \sqrt{1 - \left(\frac{m\omega}{\lambda_c\omega_c}\right)^2}, \quad \lambda_c = \frac{2\pi c}{\omega_c}
\]
\[
\Rightarrow \beta_m = k_0 \sqrt{1 - \left(\frac{m\omega}{\omega}\right)^2}
\]

The group velocity can be obtained by writing \(\beta_m^2 = k_0^2 - k_{ym}^2 = \frac{\omega^2}{c_0^2} - \left(\frac{m\pi}{d}\right)^2\). Taking the derivative in respect to \(\omega\) yields:

\[
\frac{d\beta_m^2(\omega)}{d\omega} = 2\beta_m \frac{d\beta_m}{d\omega} = \frac{2\omega}{c_0^2} \Leftrightarrow \frac{d\beta_m}{d\omega} = \frac{\omega}{c_0^2 \beta_m} = \frac{1}{v_g}.
\]
Problem 4: For the mirror waveguide we \( M_{\text{MirrorWG}} = \text{floor} \left( \frac{2d}{\lambda} \right) \) = 24 modes, corresponding to an inclination angle of \( \sin(\theta_m) = \frac{24\lambda_0}{2d} \iff \theta_24 \approx 73.4^\circ \).

For the dielectric waveguide \( M_{\text{DielectricWG}} = \text{ceil} \left( \frac{\sin(\theta_c)}{\lambda_1/2d} \right) \) = 10 modes and the angle corresponding to mode 10 is \( \theta_{10} = a \cos \left( \frac{n_2}{n_1} \right) = 14.8^\circ \).

Problem 5: We know from the book and the lecture that the NA of a step-index fiber is:

\[ NA_{\text{Step-index}} = \sqrt{n_1^2 - n_2^2} \]

We use expressions (FoP 9.1-4) for the refractive index of the graded-index fiber.

\[ n^2(r) = n_1^2 \left[ 1 - 2 \left( \frac{r}{a} \right)^p \right] \Delta \]

Now we compare with (FoP 1.3-14): \( n^2(r) = n_1^2 \left( 1 - \alpha^2 r^2 \right) \). From that we yield \( \alpha = \sqrt{n_1^2 - n_2^2} \).

We now use equation (FoP 1.3-11), which describes how a ray propagates in a GRIN medium. We chose a ray that starts at the optical axis (\( r=0 \)). We want to know the maximum angle \( \theta_c \), which remains confined in the fiber:

\[ a = \frac{\theta_c}{\alpha} \sin \left( \frac{\pi}{2} \right) \iff \theta_c = \alpha a \]

To finally calculate the NA of the graded-index fiber we assume a ray hitting the fiber on axis with an angle \( \theta \) in respect to the axis, then:

\[ NA_{\text{Graded-index}} = \sin(\theta) = n_1 \sin(\theta_c) = n_1 \alpha a = \sqrt{n_1^2 - n_2^2} \]

Note that we used the paraxial approximation \( \theta \approx \sin(\theta) \). The result is that the NAs of step-index and graded-index fibers are identical within certain approximations.

Problem 6: The fiber is single-mode, if the V parameter is <2.405. From that condition, we can calculate the cut-off wavelength \( \lambda_c=1198 \) nm. If we now look at the same fiber for a wavelength of 599 nm, the V-parameter doubles \( V=4.81 \), which are LP01, LP11, LP21, LP02. Please, note that fig. 9.2-5 in the book is wrong; it forgets about all LP_{0m} (m>1). Finally, we can look at the degeneracy of the modes; all l=0 modes have a degeneracy of 2, while all other have a degeneracy of 4. This means in total we have 12 possible modes.

Problem 7: The core radius can be obtained from the V parameter; \( a=45.9 \) μm. By numerically solving the characteristic equation (9.2-14) or by estimating from fig. 9.2-3 one can get the x-values for the different modes; \( x_{LP_{01}} \approx 2.185, x_{LP_{02}} \approx 4.997, x_{LP_{03}} \approx 7.764 \). The wavenumbers can be obtained from \( \beta_{lm} = \sqrt{k_i^2 - \frac{x_{lm}^2}{a^2}} \) (derived from 9.2-4 and 9.2-8).

\( \beta_{01} = 5.853303123 \times 10^{-6}, \beta_{02} = 5.852484711 \times 10^{-6}, \beta_{03} = 5.851053175 \times 10^{-6} \) m\(^{-1}\).
If \( V=4 \) now, the fiber radius becomes \( a \approx 18.4 \) µm. One can either estimate from fig. 9.2-8 or solve numerically for the effective refractive index \( \frac{n-n_i}{n_i-n_2} \approx 0.77 \Rightarrow n \approx 1.44377 \). From this one can calculate the phase velocity \( v_{\phi01} \approx 2.076452064 \times 10^8 \) m\(s^{-1}\) and the wave number \( \beta_{01} \approx 5.852575491 \times 10^6 \) m\(^{-1}\). The group velocity can only be obtained numerically. This is beyond the scope of these exercises, but instructive fun with Maple or Matlab. \( v_{g01} \approx 2.077654 \times 10^8 \) m\(s^{-1}\). The group velocity thus is about 0.04% slower than the phase velocity.